Multi-Resolution Video Streaming in Peer-to-Peer Networks

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Abstract

We consider multi-resolution streaming in fully-connected peer-to-peer networks, where transmission rates are constrained by arbitrarily specified upload capacities of the source and peers. We fully characterize the capacity region of rate vectors achievable with arbitrary coding, where an achievable rate vector describes a vector of throughputs of the different resolutions that can be supported by the network. We then prove that all rate vectors in the capacity region can be achieved using pure routing strategies. This shows that coding has no capacity advantage over routing in this scenario.

I. INTRODUCTION

We consider multi-resolution streaming in a heterogeneous peer-to-peer setting, where peers have different upload capacities and demand an information stream at different resolutions. The information stream is layered, such as in Scalable Video Coding [1], which generates a base video layer and a number of enhancement layers that depend on the base layer and all lower layers.

We assume a fully-connected overlay network in which transmission rates are constrained by the upload capacity of the source and each peer, a model introduced in [2] to capture the most important constraints in peer-to-peer networks. A problem instance is defined by specifying the number of layers demanded by each peer and the upload capacity constraints of the source and each peer. Our goal is to find the capacity region of achievable rate vectors, where an achievable rate vector describes a vector of throughputs of the different resolutions that can be supported by the network.

Solutions can be classified as follows. Inter-session coding solutions are the most general, allowing coding across information from different sessions (i.e. layers). Intra-session coding solutions restrict coding to occur only within each session. Routing solutions allow only replication and forwarding of information at each node. Intra-session coding corresponds to independent multicast network coding for each layer, for which the capacity region is given by a linear program. In contrast, characterizing inter-session coding capacity, which corresponds to the information theoretic capacity, is open for general networks.

Related work by Chiu et al. [3] studies the special case of a single resolution. That case corresponds to a single multicast, and [3] shows that network coding is not needed to achieve capacity. In [4], Ponec et al. consider the multi-resolution case restricted to intra-session coding, showing that intra-session coding does not improve the capacity region over routing. A different objective of minimizing average finish times for file download was studied in [5], [6].

In this paper we provide a complete characterization of the capacity region of feasible rate vectors achievable with arbitrary (inter or intra-session) coding, and show that the entire capacity region can be achieved with routing.

II. PROBLEM DEFINITION

A peer to peer network is modeled as a complete directed graph with a single source node p_0 and $k \geq 1$ peer nodes $\{p_0, p_1, \ldots, p_k\}$. The upload capacities of nodes $p_0, \ldots, p_k$ are $C_0, \ldots, C_k$ respectively.

Information originates at the source node and is distributed to the peers, which help the distribution process by uploading information to other peers. Coding may occur at the source and peers.

Let n be the number of different resolutions in a layered data stream. We denote by $x_1, \ldots, x_n$ the data streams corresponding to the different layers, such that the jth resolution corresponds to $\{x_1, x_2, \ldots, x_j\}$. The rate of x_j is denoted by L_j. For simplicity, we assume that the upload capacities $C_0, \ldots, C_k$ and the data rates $L_1, \ldots, L_n$ are integers, which can be approached arbitrarily closely by scaling the unit appropriately.

We are given nested sets $X_1, \ldots, X_n$ specifying the demands:

$$X_j = \{p_i | p_i \ demands \ x_j\}.$$

We also define $X_{n+1} = \{p_0\}$, so we have

$$\{p_0\} = X_{n+1} \subseteq X_n \subseteq \ldots \subseteq X_1 = \{p_0, p_1, \ldots, p_k\}.$$

For all $S_1, S_2 \subseteq X_1$, $S_1 \to S_2$ is defined as the set of all links coming from S_1 and going to S_2. We also write $p_i \to p_j$ instead of $\{p_i\} \to \{p_j\}$ for brevity. The constraints on a successful transmission scheme are as follows:

i) Each outgoing link of the source p_0 is a function of $x_1, \ldots, x_n$:

$$\forall p_i \in X_1, \ H(p_0 \to p_i | x_1, \ldots, x_n) = 0.$$

ii) Each outgoing link of $p_i \in X_1 \setminus \{p_0\}$ is a function of incoming links:

$$\forall p_i, p_j \in X_1 \setminus \{p_0\}, \ H(p_i \to p_j | p_0 \to p_i, \ldots, p_n \to p_i) = 0.$$

We assume that $H(p_i \to p_i) = 0$ without loss of optimality.

iii) Each peer $p_i \in X_1$ can transmit at rate at most C_i:

$$\forall p_i \in X_1 \ C_i \geq \sum_{p_j \in X_1} H(p_i \to p_j)$$

iv) Each peer $p_i \in X_j \setminus \{p_0\}$ is able to decode x_j from its received information:

$$I(X_1 \to p_i; x_j) = H(x_j) = L_j.$$

III. APPROACH

In this section we provide some intuition for our approach. A first observation is that total upload capacity should be greater than the total rate of data which has to be delivered:

$$\sum_{i=0} C_i \geq \sum_{i=1}^{n} |X_i| L_i. \tag{1}$$

This condition is necessary but not sufficient. The following sequence of lemmas leads to a sufficient condition for a rate vector to be achievable. The proofs of lemmas in this and the next section can be found in the Appendix.

Lemma 1. *Let k and C_0 be positive integers and $C_1, C_2, \ldots, C_k$ be nonnegative integers such that*

$$C_0 + \sum_{i=1}^{k} C_i = k$$

Then there exists a directed tree rooted at v_0 with vertices $v_1, \ldots, v_k$ such that

$$outdeg(v_0) = C_0,$$

$$\forall i \in \{1, \ldots, k\} \ outdeg(v_i) = C_i, \ indeg(v_i) = 1.$$

Lemma 2. *Data x with rate L can be transmitted to peers $p_1, \ldots, p_k$ by using source capacity C_0 and peer upload capacities $C_1, \ldots, C_k$ if*

$$C_0 \geq L \ and$$

$$\sum_{i=0}^{k} C_i \geq kL.$$

Lemma 3. *Given the sets of peers $X_1, X_2, \ldots, X_n$ and upload capacities $C_0, C_1, \ldots, C_k$, the rate vector $(L_1, L_2, \ldots, L_n)$ is achievable if for every $j \in \{1, \ldots, n\}$*

$$\sum_{p_i \in X_j} C_i \geq \sum_{i=1}^{j-1} L_i + \sum_{i=j}^{n} |X_i| L_i \tag{2}$$

and

$$C_0 \geq \sum_{i=1}^{n} L_i. \tag{3}$$

Intuitively, if one of the inequalities in Lemma 3, say the jth one, does not hold, this means that the nodes in set X_j cannot handle the transmission of data layers x_j through x_n. Hence, some peers from the set $X_1 \setminus X_j$ need to help in transmitting those data layers, necessitating some additional capacity for transmitting this data to peers in $X_1 \setminus X_j$ which do not themselves demand it. This requires additional capacity beyond that given in (1).

To characterize this explicitly, it is useful to define the margin of the jth inequality:

$$N_j = \sum_{i=j}^{n} |X_i| L_i + \sum_{i=1}^{j-1} L_i - \sum_{p_i \in X_j} C_i.$$

For completeness we also define the $(n+1)$-th margin N_{n+1} as zero. The capacity region derived in the next section is stated in terms of these margins. In fact, not all of them, but a special subset of them, will be used. This subset is defined as follows:

Definition 1. *For a finite sequence $\{a_n\} = a_1, \ldots, a_s$, the dominant subsequence of $\{a_n\}$ is the subsequence $\{a_{i_n}\} = a_{i_1}, \ldots, a_{i_h}$ defined by*
   i) $i_h = s$
   ii) i_j is the greatest index such that $i_j < i_{j+1}$ and $a_{i_j} > a_{i_{j+1}}$.

IV. CONVERSE BOUND FOR CAPACITY REGION

In this section, we present a converse bound on the capacity region, which is shown to be tight in the following section.

Theorem 1. *Given the sets of peers $X_1, X_2, \ldots, X_n$ and upload capacities $C_0, C_1, \ldots, C_k$, if the rate vector $(L_1, L_2, \ldots, L_n)$ is achievable by any coding scheme, then*

$$\sum_{p_i \in X_1} C_i \geq \sum_{i=1}^{n} |X_i| L_i + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1}$$

where $N_{d_1}, \ldots, N_{d_{h+1}}$ is the dominant subsequence of $N_1, \ldots, N_{n+1}$.

   Proof: For a resolution x_j and a peer $p_i \in X_j$ we have, from property iv in Section II,

$$H(x_j) = I(X_1 \to p_i; x_j) = I(X_1 \setminus X_j \to p_i, X_j \to p_i; x_j).$$

If we view $X_1 \setminus X_j$ as a supernode, outgoing links should be functions of incoming links, since peers in the set $X_1 \setminus X_j$ do not create additional data besides incoming data (property ii). Hence, links in set $X_1 \setminus X_j \to p_i$ are completely dependent on links in set $X_j \to X_1 \setminus X_j$. Then, we may write:

$$H(x_j) = I((X_1 \setminus X_j) \to p_i, X_j \to p_i; x_j) \leq$$
$$I(X_j \to (X_1 \setminus X_j), X_j \to p_i; x_j) \leq H(x_j)$$
$$\Rightarrow \quad H(X_j) = I(X_j \to (X_1 \setminus X_j); x_j)$$
$$+ I(X_j \to p_i; x_j | X_j \to (X_1 \setminus X_j)).$$

Summing this for all peers in X_j yields

$$|X_j| H(X_j) = |X_j| I(X_j \to (X_1 \setminus X_j); x_j) +$$
$$\sum_{p_i \in X_j} I(X_j \to p_i; x_j | X_j \to (X_1 \setminus X_j)).$$

By rearranging this, we can obtain

$$I(X_j \to (X_1 \setminus X_j); x_j) = \frac{1}{|X_j| - 1} [|X_j| H(x_j) -$$
$$I(X_j \to (X_1 \setminus X_j); x_j) -$$

$$\sum_{p_i \in X_j} I(X_j \to p_i; x_j | X_j \to (X_1 \setminus X_j))]. \tag{4}$$

The left hand side of this equation can be replaced by parameters which are independent from the transmission scheme by using the following lemma:

Lemma 4. $\sum_{j=1}^{n} I(X_j \to (X_1 \setminus X_j); x_j) \leq \sum_{p_i \in X_j} C_i - \sum_{j=1}^{n} |X_j| H(x_j)$.

By using Equation (4) and Lemma 4, we can obtain:

$$\sum_{p_i \in X_j} C_i \geq \sum_{j=1}^{n} |X_j| H(x_j) + \sum_{j=1}^{n} \frac{1}{|X_j| - 1} [|X_j| H(x_j)$$
$$- I(X_j \to (X_1 \setminus X_j); x_j)$$
$$- \sum_{p_i \in X_j} I(X_j \to p_i; x_j | X_j \to (X_1 \setminus X_j))]. \tag{5}$$

Now define A_j as

$$A_j = \sum_{l=j}^{n} |X_l| H(x_l) - \sum_{l=j}^{n} [I(X_l \to (X_1 \setminus X_l); x_l) +$$
$$\sum_{p_i \in X_l} I(X_l \to p_i; x_l | X_l \to (X_1 \setminus X_l))].$$

For the sake of completeness, define also $A_{n+1} = 0$. Putting this into Inequality (5), we obtain:

$$\sum_{p_i \in X_j} C_i \geq \sum_{j=1}^{n} |X_j| H(x_j) + \sum_{j=1}^{n} \frac{A_j - A_{j+1}}{|X_j| - 1}. \tag{6}$$

We also have, from Equation (4):

$$A_j - A_{j+1} = (|X_j| - 1) I(X_j \to (X_1 \setminus X_j); x_j) \geq 0. \tag{7}$$

Note that the A_j values are determined by the transmission scheme. To obtain a bound which is independent from transmission scheme, we use the following lemma:

Lemma 5. $A_j \geq N_j$.

Let us examine the last sum in (6):

$$\sum_{j=1}^{n} \frac{A_j - A_{j+1}}{|X_j| - 1} = \sum_{j=1}^{d_1 - 1} \frac{A_j - A_{j+1}}{|X_j| - 1} + \sum_{j=d_1}^{d_2 - 1} \frac{A_j - A_{j+1}}{|X_j| - 1} + \ldots +$$
$$\sum_{j=d_h}^{n} \frac{A_j - A_{j+1}}{|X_j| - 1}.$$

Using inequality (7),

$$0 + \sum_{j=d_1}^{d_2-1} \frac{A_j - A_{j+1}}{|X_{d_1}| - 1} + \ldots + \sum_{j=d_h}^{n} \frac{A_j - A_{j+1}}{|X_{d_h}| - 1}$$

$$= \frac{A_{d_1}}{|X_{d_1}| - 1} + A_{d_2}\left(\frac{1}{|X_{d_2}| - 1} - \frac{1}{|X_{d_1}| - 1}\right) + \ldots + A_{d_h}\left(\frac{1}{|X_{d_h}| - 1} - \frac{1}{|X_{d_{h-1}}| - 1}\right)$$

$$\geq \frac{N_{d_1}}{|X_{d_1}| - 1} + N_{d_2}\left(\frac{1}{|X_{d_2}| - 1} - \frac{1}{|X_{d_1}| - 1}\right) + \ldots + N_{d_h}\left(\frac{1}{|X_{d_h}| - 1} - \frac{1}{|X_{d_{h-1}}| - 1}\right)$$

$$= \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1}.$$

The last inequality is due to Lemma 5 . By putting this result in (6), we obtain:

$$\sum_{p_i \in X_1} C_i \geq \sum_{i=1}^{n} |X_i| L_i + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1}$$

■

V. A ROUTING SCHEME THAT ACHIEVES THE CAPACITY REGION

In this section, we give a transmission scheme using multicast routing trees that achieves the bound in Theorem 1.

Theorem 2. *Given the sets of peers $X_1, X_2, \ldots, X_n$ and upload capacities $C_0, C_1, \ldots, C_k$, the rate vector $(L_1, L_2, \ldots, L_n)$ is achievable by routing if*

$$C_0 \geq \sum_{i=1}^{n} L_i, \tag{8}$$

$$\sum_{p_i \in X_1} C_i \geq \sum_{i=1}^{n} |X_i| L_i + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1} \tag{9}$$

where $N_{d_1}, \ldots, N_{d_{h+1}}$ is the dominant subsequence of $N_1, \ldots, N_{n+1}$.

Proof: The proof is by induction on the number of inequalities from Lemma 3 which are not satisfied. For this purpose let us define the set

$$I = \{i | N_i > 0\}.$$

We will show that we can reduce the size of I by at least one, by using some amount of capacity, such that the residual system also satisfies (8) and (9).

The base case is the one where I is the empty set, i.e. all N_j values are less than or equal to zero and it is examined in Lemma 3. Let m and M denote the minimum and maximum elements of I, respectively. The following lemma is essential for our method:

Lemma 6. $\forall p_i \in X_1 \setminus X_m$, $\exists C_{iM}$ with $0 \leq C_{iM} \leq C_i$ such that

$$\sum_{p_i \in X_j \setminus X_m} C_{iM} \leq N - N_j$$

$\forall j \in \{1, 2, \ldots, m-1\}$ and

$$\sum_{p_i \in X_1 \setminus X_m} C_{iM} = \frac{|X_M| N}{|X_M| - 1}$$

where $N = min(N_m, N_M)$.

Now, for each $p_i \in X_1 \setminus X_m$ let us choose C_{iM} as in Lemma 6. Then we have

$$\sum_{p_i \in X_1 \setminus X_m} C_{iM} = \frac{|X_M|N}{|X_M| - 1} \tag{10}$$

where $N = \min(N_m, N_M)$. Let us also define

$$C_{0M} = \frac{N}{|X_M| - 1}.$$

Now take a rate-$\frac{N}{|X_M|-1}$ portion of x_M, called s. Then, using equality (10), the rate of s is given by

$$\frac{N}{|X_M| - 1} = \sum_{p_i \in X_1 \setminus X_m} \frac{C_{iM}}{|X_M|}.$$

Hence we can divide s into $|X_1 \setminus X_m|$ portions s_i corresponding to peers $p_i \in X_1 \setminus X_m$, where the rate of portion s_i is given by $\frac{C_{iM}}{|X_M|}$. To each peer $p_i \in X_1 \setminus X_m$, send s_i from the source. This consumes C_{0M} amount of capacity of the source. Then, from each $p_i \in X_1 \setminus X_m$, send s_i to the peers in X_M. This consumes C_{iM} amount of capacity of each $p_i \in X_1 \setminus X_m$. In this way, transmission of portion s is completed. After the procedure, we have residual capacities

$$C_i' = \begin{cases} C_i - C_{iM} & if \ p_i \in X_1 \setminus X_m \\ C_i & if \ p_i \in X_m \setminus \{p_0\} \\ C_0 - \frac{N}{|X_M|-1} & if \ p_i = p_0. \end{cases}$$

and residual data rates

$$L_i' = \begin{cases} L_i - \frac{N}{|X_M|-1} & if \ i = M \\ L_i & otherwise. \end{cases}$$

The N_i values are updated accordingly. Denoting the updated value of N_j by N_j', we calculate it differently for three cases:

i) If $j < m$:

$$
\begin{aligned}
N_j' &= \sum_{i=j}^{n} |X_i| L_i' - C_0' + \sum_{i=1}^{j-1} L_i' - \sum_{p_i \in X_j \setminus \{p_0\}} C_i' \\
&= \sum_{i=j}^{n} |X_i| L_i - \frac{|X_M|N}{|X_M| - 1} - C_0 + \frac{N}{|X_M| - 1} + \sum_{i=1}^{j-1} L_i \\
&\quad - \sum_{p_i \in X_j \setminus \{p_0\}} C_i + \sum_{p_i \in X_j \setminus X_m} C_{iM} \\
&= (\sum_{i=j}^{n} |X_i| L_i + \sum_{i=1}^{j-1} L_i - \sum_{p_i \in X_j} C_i) \\
&\quad - (\frac{|X_M|N}{|X_M| - 1} - \frac{N}{|X_M| - 1}) + \sum_{p_i \in X_j \setminus X_m} C_{iM} \\
&= N_j - N + \sum_{p_i \in X_j \setminus X_m} C_{iM}.
\end{aligned}
$$

By the choice of C_{iM} values, using Lemma 6, we have

$$N_j' = N_j - N + \sum_{p_i \in X_j \setminus X_m} C_{iM} \leq 0 \tag{11}$$

ii) If $m \leq j \leq M$:

$$N'_j = \sum_{i=j}^{n} |X_i| L'_i - C'_0 + \sum_{i=1}^{j-1} L'_i - \sum_{p_i \in X_j \setminus \{p_0\}} C'_i$$

$$= \sum_{i=j}^{n} |X_i| L_i - \frac{|X_M| N}{|X_M| - 1} - C_0 + \frac{N}{|X_M| - 1}$$

$$+ \sum_{i=1}^{j-1} L_i - \sum_{p_i \in X_j \setminus \{p_0\}} C_i$$

$$\Rightarrow \quad N'_j = N_j - N. \tag{12}$$

iii) If $j > M$:

$$N'_j = \sum_{i=j}^{n} |X_i| L'_i - C'_0 + \sum_{i=1}^{j-1} L'_i - \sum_{p_i \in X_j \setminus \{p_0\}} C'_i$$

$$= \sum_{i=j}^{n} |X_i| L_i - C_0 + \frac{N}{|X_M| - 1} + \sum_{i=1}^{j-1} L_i$$

$$- \frac{N}{|X_M| - 1} - \sum_{p_i \in X_j \setminus \{p_0\}} C_i$$

$$\Rightarrow \quad N'_j = N_j < 0. \tag{13}$$

Now let us examine the dominant subsequence $N'_{d'_1}, \ldots, N'_{d'_{h'+1}}$ of $N'_1, \ldots, N'_{n+1}$. Since N'_{n+1} is zero by definition, $N'_{d'_{h'+1}}$ is also zero. Therefore for all $i \in \{1, \ldots, h'\}$

$$N'_{d'_i} > N'_{d'_{h'+1}} = 0 \Rightarrow m \leq d'_i \leq M.$$

But the order of N'_i values for $m \leq i \leq M$ is the same as the order of N_i values for $m \leq i \leq M$ since $N'_j = N_j - N$. Also we know that for all $i \in \{1, \ldots, h\}$

$$m \leq d_i \leq M.$$

Noting that $N_{d_h} = N_M$, we have two cases:

$$(N'_{d'_1}, \ldots, N'_{d'_{h'+1}}) = (N_{d_1} - N, \ldots, N_{d_h} - N, 0)$$

if $N = N_m < N_M$, and

$$(N'_{d'_1}, \ldots, N'_{d'_{h'+1}}) = (N_{d_1} - N, \ldots, N_{d_h - 1} - N, 0)$$

if $N = N_M \leq N_m$.

These two cases can be considered as one since even if N is equal to N_M, we can consider as $N'_{d'_{h'}} = 0 = N_{d_h} - N$ so that it does not affect inequality (9). Hence we can write:

$$(N'_{d'_1}, \ldots, N'_{d'_{h'}}, N'_{d'_{h'+1}}) = (N_{d_1} - N, \ldots, N_{d_h} - N, 0).$$

Now let us calculate the left hand side of the inequality (9) with updated values:

$$
\sum_{p_i \in X_1} C_i' = C_0 - \frac{N}{|X_M| - 1} + \sum_{p_i \in X_1 \setminus \{p_0\}} C_i - \sum_{p_i \in X_1 \setminus X_m} C_{iM}
$$

$$
= \sum_{p_i \in X_1} C_i - \frac{N}{|X_M| - 1} - \frac{|X_M| N}{|X_M| - 1}
$$

$$
\geq \sum_{i=1}^{n} |X_i| L_i + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1} - \frac{(|X_M| + 1) N}{|X_M| - 1}
$$

$$
= \sum_{i=1}^{n} |X_i| L_i' + \frac{|X_M| N}{|X_M| - 1} + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1} - \frac{(|X_M| + 1) N}{|X_M| - 1}
$$

$$
= \sum_{i=1}^{n} |X_i| L_i' + \sum_{i=1}^{h-1} \frac{(N_{d_i} - N) - (N_{d_{i+1}} - N)}{|X_{d_i}| - 1} + \frac{N_{d_h} - N}{|X_{d_h}| - 1}
$$

$$
= \sum_{i=1}^{n} |X_i| L_i' + \sum_{i=1}^{h-1} \frac{N_{d_i'}' - N_{d_i'+1}'}{|X_{d_i}| - 1} + \frac{N_{d_h'}' - 0}{|X_{d_h}| - 1}
$$

$$
= \sum_{i=1}^{n} |X_i| L_i' + \sum_{i=1}^{h} \frac{N_{d_i'}' - N_{d_i'+1}'}{|X_{d_i}| - 1}.
$$

This shows that inequality (9) is preserved after the procedure. Inequality (8) is also preserved since

$$
C_0' = C_0 - \frac{N}{|X_M| - 1} \geq \sum_{i=1}^{n} L_i - \frac{N}{|X_M| - 1} = \sum_{i=1}^{n} L_i'.
$$

Now let us look at the updated version I' of I. From (11), (12) and (13) we know that $I' \subseteq I$. If $N = N_m$, then $N_m' = N_m - N = 0 \Rightarrow m \notin I' \Rightarrow |I'| \leq |I| - 1$. Similarly if $N = N_M$, then $M \notin I' \Rightarrow |I'| \leq |I| - 1$. Finally we can say that after reducing the rate of x_M by $\frac{N}{|X_M|-1}$, inequalities (8) and (9) are still correct and number of inequalities from Lemma 3 is reduced by at least one. Hence, by the induction hypothesis, we can complete transmission of the remaining data. This completes the proof. ∎

Combining Theorem 1 and Theorem 2 with the addition of trivial condition $C_0 \geq \sum_{i=1}^{n} L_i$, we obtain the exact capacity region:

Corollary 1. *Given the sets of peers* $X_1, X_2, \ldots, X_n$ *and upload capacities* $C_0, C_1, \ldots, C_k$, *the rate vector* $(L_1, L_2, \ldots, L_n)$ *is achievable if and only if the following inequalities hold*

$$
C_0 \geq \sum_{i=1}^{n} L_i
$$

$$
\sum_{p_i \in X_1} C_i \geq \sum_{i=1}^{n} |X_i| L_i + \sum_{i=1}^{h} \frac{N_{d_i} - N_{d_{i+1}}}{|X_{d_i}| - 1}
$$

where $N_{d_1}, \ldots, N_{d_{h+1}}$ *is the dominant subsequence of* $N_1, \ldots, N_{n+1}$. *Furthermore, the capacity region is achievable using routing.*

VI. Conclusion

We have characterized the capacity region of achievable rates for multi-resolution streaming in peer-to-peer networks with upload capacity constraints, and shown that this region can be achieved by routing. This represents a new class of non-multicast network problems for which we have a capacity characterization. Although coding is not needed to achieve capacity in this scenario, it can nevertheless be useful in scenarios with losses or without centralized control.

Proof of Lemma 1:

By Induction on k

  base case: $k = C_0$

$$C_0 + \sum_{i=1}^{k} C_i = k = C_0 = outdeg(v_0) \ \Rightarrow \ \sum_{i=1}^{k} C_i = 0 \Rightarrow$$

$$\forall i \in \{1, \ldots, k\} \ outdeg(v_i) = C_i = 0$$

Then a directed tree rooted at v_0 with k leaves, namely $v_1, v_2, \ldots, v_k$, gives us the desired tree.

Now we can assume that $C_1 \leq \ldots \leq C_{k-1} \leq C_k$ without loss of generality. Then $C_k \geq 1$, since it would be the base case otherwise. By inductive hypothesis on $k - 1$, there exists a directed tree rooted at v_0 with vertices $v_1, \ldots, v_{k-1}$ such that

$$outdeg(v_0) = C_0,$$

$$\forall i \in \{1, \ldots, k-2\} \ outdeg(v_i) = C_i, \ indeg(v_i) = 1, \ and$$

$$outdeg(v_{k-1}) = (C_{k-1} + C_k - 1), \ indeg(v_{k-1}) = 1$$

since

$$C_0 + C_1 + \ldots + C_{k-2} + (C_{k-1} + C_k - 1) = k - 1.$$

Now, we add a new vertex, namely v_k, to this tree as an isolated node and then apply the following two changes in order:

1) Remove arbitrarily selected C_k outgoing edges from v_{k-1} where $C_k \geq 1$, then add a new directed edge from v_{k-1} to v_k, hence we now have $outdeg(v_{k-1}) = C_{k-1}$ and $indeg(v_k) = 1$.

2) Let D be the set of nodes that are disconnected from v_{k-1} in the first step. Note that D is not empty since $|D| = C_k \geq 1$. Now add a new directed edge from v_k to each node in set D. Hence, $outdeg(v_k) = C_k$.

By noting that the number of outgoing/incoming edges from/to the vertices in set D have not been changed, the resulting directed tree rooted at v_0 with vertices $v_1, v_2, \ldots, v_k$ is exactly the one which is desired. This completes the induction and the proof of the lemma. ∎

Proof of Lemma 2:

If C_0, C_i's or L are not integers we can divide the unit length to least common multiple of denominators to make them so. Thus, we will assume that they are integers in this proof.

We will do induction on L. Base case $L = 1$ is due to lemma 1 . Now let us assume that the claim is true for $L - 1$. Since

$$\sum_{i=0}^{k} C_i \geq kL,$$

we can find integers $\overline{C_0}, \overline{C_1}, \ldots, \overline{C_k}$ such that

$$L - 1 \leq \overline{C_0} \leq C_S - 1$$

$$0 \leq \overline{C_1} \leq C_1$$

$$\vdots$$

$$0 \leq \overline{C_k} \leq C_k$$

$$\sum_{i=0}^{k} \overline{C_i} = k(L - 1)$$

If we divide the data x into L unit length parts $x_1, x_2, \ldots, x_L$, by the induction hypothesis we can complete the distribution of parts x_2 through x_L by using $\overline{C_0}, \overline{C_1}, \ldots, \overline{C_k}$ capacities. Then we will have capacities $C_0 - \overline{C_0}, C_1 - \overline{C_1}, \ldots, C_k - \overline{C_k}$ such that

$$C_0 - \overline{C_0} \geq 1$$

$$\sum_{i=0}^{k}(C_i - \overline{C_i}) \geq k.$$

Hence, we can construct a routing tree for x_1 by using the Lemma 1, thereby finishing the distribution of complete data $x = x_1 x_2 \ldots x_L$. ∎

Proof of Lemma 3:
We will do induction on the number of resolutions n.

Base Case: $n = 1$ is stated in Lemma 2.

Inductive Step: For $j = n$ in (2), we have

$$\sum_{p_i \in X_n} C_i - \sum_{i=1}^{n-1} L_i \geq |X_n| L_n. \tag{14}$$

Since we have inequalities (3) and (14), there exist $0 \leq \overline{C_i} \leq C_i$ for each $p_i \in X_n \setminus \{p_0\}$ and $L_n \leq \overline{C_0} \leq C_0 - \sum_{i=1}^{n-1} L_i$ for source such that

$$\sum_{p_i \in X_n} \overline{C_i} - \sum_{i=1}^{n-1} L_i = |X_n| L_n. \tag{15}$$

Since we have (15) and $\overline{C_0} - \sum_{i=1}^{n-1} L_i \geq L_n$, by Lemma 2, each peer in set X_n can receive the resolution x_n completely by using $\overline{C_i}$ capacity from each $p_i \in X_n$ including source. After completing the transmission of the resolution x_n, each peer $p_i \in X_n$ needs to receive only the resolutions $x_1, x_2, \ldots, x_{n-1}$ same as all the peers in X_{n-1}.

If we denote the remaining capacity of a peer p_i with C_i' and of source with C_0', we may write

$$C_i' = \begin{cases} C_i & if \ p_i \in X_1 \setminus X_n \\ C_i - \overline{C_i} & if \ p_i \in X_n \end{cases}$$

Then, for each $j \in \{1, \ldots, n-1\}$, following is true by using (2) and (15):

$$\sum_{p_i \in X_j} C_i' - \sum_{i=1}^{j-1} L_i = -\sum_{i=1}^{j-1} L_i + \sum_{p_i \in X_j \setminus X_n} C_i + \sum_{p_i \in X_n}(C_i - \overline{C_i})$$

$$= -\sum_{i=1}^{j-1} L_i + \sum_{p_i \in X_j} C_i - |X_n| L_n$$

$$\geq \sum_{i=j}^{n} |X_i| L_i - |X_n| L_n = \sum_{i=j}^{n-1} |X_i| L_i.$$

Hence we have, for resolutions $x_1, \ldots, x_{n-1}$, for every $j \in \{1, \ldots, n-1\}$

$$\sum_{p_i \in X_j} C_i' \geq \sum_{i=1}^{j-1} L_i + \sum_{i=j}^{n-1} |X_i| L_i$$

and

$$C_0' \geq \sum_{i=1}^{n-1} L_i.$$

Finally we can claim that these capacities are enough for transmission of $x_1, \ldots, x_{n-1}$ by the inductive hypothesis, thereby completing all transmission. ∎

Proof of Lemma 4:
For simplicity, we use the notation

$$H(S) = H(X_1 \to S)$$

to represent the entropy of all information $X_1 \to S$ recieved by set S and use analogous notation for other information theoretic quantities. Then

$$\sum_{j=1}^{n} I(X_j \to (X_1 \setminus X_j); x_j) \le \sum_{j=1}^{n} I(X_1 \setminus X_j; x_j) = \sum_{j=1}^{n} \sum_{i=1}^{j-1} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

Let us examine the inner sum:

$$\sum_{j=i+1}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j) = H(X_i \setminus X_{i+1}) - H(X_i \setminus X_{i+1} | x_{i+1}) + \sum_{j=i+2}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

$$= H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) + I(X_i \setminus X_{i+1}; x_1, \ldots, x_i) - H(X_i \setminus X_{i+1} | x_1, \ldots, x_{i+1}) - I(X_i \setminus X_{i+1}; x_1, \ldots, x_i | x_{i+1})$$

$$+ \sum_{j=i+2}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

Since $x_1, \ldots, x_i$ are independent from x_{i+1} and peers in set $X_i \setminus X_{i+1}$ can decode $x_1, \ldots, x_i$:

$$= H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) + H(x_1, \ldots, x_i) - H(X_i \setminus X_{i+1} | x_1, \ldots, x_{i+1}) - H(x_1, \ldots, x_i)$$

$$+ \sum_{j=i+2}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

$$= H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) - H(X_i \setminus X_{i+1} | x_1, \ldots, x_{i+1}) + \sum_{j=i+2}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

$$= H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) - H(X_i \setminus X_{i+1} | x_1, \ldots, x_{i+1}) + \sum_{j=i+2}^{n} [H(X_i \setminus X_{i+1} | X_{i+1} \setminus X_j) - H(X_i \setminus X_{i+1} | x_j, X_{i+1} \setminus X_j)]$$

$$= H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) - H(X_i \setminus X_{i+1} | x_1, \ldots, x_{i+1}) + H(X_i \setminus X_{i+1} | X_{i+1} \setminus X_{i+2})$$

$$- \sum_{j=i+2}^{n} [H(X_i \setminus X_{i+1} | x_j, X_{i+1} \setminus X_j) - H(X_i \setminus X_{i+1} | X_{i+1} \setminus X_{j+1})] - H(X_i \setminus X_{i+1} | x_n, X_{i+1} \setminus X_n)$$

$$\le H(X_i \setminus X_{i+1} | x_1, \ldots, x_i) \le \sum_{p_t \in X_i \setminus X_{i+1}} H(p_t | x_1, \ldots, x_i) = \sum_{p_t \in X_i \setminus X_{i+1}} [H(p_t) - I(p_t; x_1, \ldots, x_i)]$$

$$= \sum_{p_t \in X_i \setminus X_{i+1}} H(p_t) - |X_i \setminus X_{i+1}| \sum_{j=1}^{i} H(x_j)$$

Hence we can write

$$\sum_{j=1}^{n} I(X_j \to (X_1 \setminus X_j); x_j) \le \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I(X_i \setminus X_{i+1}; x_j | X_{i+1} \setminus X_j)$$

$$\le \sum_{i=1}^{n-1} \left(\sum_{p_t \in X_i \setminus X_{i+1}} H(p_t) - |X_i \setminus X_{i+1}| \sum_{j=1}^{i} H(x_j) \right) = \sum_{p_i \in X_1 \setminus \{p_0\}} H(p_i) - \sum_{j=1}^{n} |X_j| H(x_j)$$

$$= \sum_{p_i \in X_1 \setminus \{p_0\}} H(X_1 \to p_i) - \sum_{j=1}^{n} |X_j| H(x_j) \le \sum_{p_i \in X_1 \setminus \{p_0\}} \sum_{p_l \in X_1} H(p_l \to p_i) - \sum_{j=1}^{n} |X_j| H(x_j)$$

$$= \sum_{p_l \in X_1} \sum_{p_i \in X_1 \setminus \{p_0\}} H(p_l \to p_i) - \sum_{j=1}^{n} |X_j| H(x_j) \le \sum_{p_l \in X_1} C_l - \sum_{j=1}^{n} |X_j| H(x_j)$$

Last inequality is due to *property iii*. ∎

Proof of Lemma 5:

$$A_j \geq N_j \Leftrightarrow$$

$$\sum_{l=j}^{n} |X_l| H(x_l) - \sum_{l=j}^{n} \left(I(X_l \rightarrow (X_1 \setminus X_l); x_l) + \sum_{p_i \in X_l} I(X_l \rightarrow p_i; x_l | X_l \rightarrow (X_1 \setminus X_l)) \right)$$

$$\geq \sum_{l=j}^{n} |X_l| L_l + \sum_{l=1}^{j-1} L_l - \sum_{p_i \in X_j} C_i.$$

Then, all we have to prove is , putting $H(x_l)$ in place of L_l,

$$\sum_{p_i \in X_j} C_i \geq \sum_{l=1}^{j-1} H(x_l) + \sum_{l=j}^{n} I(X_l \rightarrow (X_1 \setminus X_l); x_l) + \sum_{l=j}^{n} \sum_{p_i \in X_l} I(X_l \rightarrow p_i; x_l | X_l \rightarrow (X_1 \setminus X_l))$$

Following lemma will help us

Sublemma 1. *For each $j, m \geq j$ and variables A, B*

$$\sum_{l=j}^{m} I(A; x_l | X_l \rightarrow X_1 \setminus X_l, B) \leq H(A).$$

Proof:

$$\sum_{l=j}^{m} I(A; x_l | X_l \rightarrow X_1 \setminus X_l, B) =$$

$$\sum_{l=j}^{m} H(A | X_l \rightarrow X_1 \setminus X_l, B) - \sum_{l=j}^{m} H(A | x_l, X_l \rightarrow X_1 \setminus X_l, B)$$

$$= H(A | X_j \rightarrow X_1 \setminus X_j, B) + \sum_{l=j+1}^{m} H(A | X_l \rightarrow X_1 \setminus X_l, B)$$

$$- \sum_{l=j}^{m-1} H(A | x_l, X_l \rightarrow X_1 \setminus X_l, B) - H(A | x_l, X_m \rightarrow X_1 \setminus X_m, B)$$

$$= H(A | X_j \rightarrow X_1 \setminus X_j, B) + \sum_{l=j}^{m-1} H(A | X_{l+1} \rightarrow X_1 \setminus X_{l+1}, B)$$

$$- \sum_{l=j}^{m-1} H(A | x_l, X_l \rightarrow X_1 \setminus X_l, B) - H(A | x_l, X_m \rightarrow X_1 \setminus X_m, B)$$

$$\leq H(A) + \sum_{l=j}^{m-1} [H(A | X_{l+1} \rightarrow X_1 \setminus X_{l+1}, B) - H(A | x_l, X_l \rightarrow X_1 \setminus X_l, B)].$$

Let us examine $X_l \rightarrow X_1 \setminus X_l$:

$$X_1 \rightarrow X_1 \setminus X_l = (X_l \rightarrow X_1 \setminus X_l) \cup (X_1 \setminus X_l \rightarrow X_1 \setminus X_l).$$

With a similar idea used in the proof of Theorem 2, we can claim that $X_1 \setminus X_l \rightarrow X_1 \setminus X_l$ is completely dependent on $X_l \rightarrow X_1 \setminus X_l$. Thus $X_1 \rightarrow X_1 \setminus X_l$ is also completely dependent on $X_l \rightarrow X_1 \setminus X_l$. Since

$X_l \to X_1 \setminus X_l \subseteq X_1 \to X_1 \setminus X_l$, reverse is also true. Hence, we can use sets $X_l \to X_1 \setminus X_l$ and $X_1 \to X_1 \setminus X_l$ interchangeably in information theoretic expressions. Then,

$$\sum_{l=j}^{m} I(A; x_l | X_l \to X_1 \setminus X_l, B) \leq$$

$$H(A) + \sum_{l=j}^{m-1} [H(A|X_{l+1} \to X_1 \setminus X_{l+1}, B) - H(A|x_l, X_l \to X_1 \setminus X_l, B)] =$$

$$H(A) + \sum_{l=j}^{m-1} [H(A|X_1 \to X_1 \setminus X_{l+1}, B) - H(A|x_l, X_1 \to X_1 \setminus X_l, B)] =$$

$$H(A) + \sum_{l=j}^{m-1} [H(A|X_1 \to X_l \setminus X_{l+1}, X_1 \to X_1 \setminus X_l, B) - H(A|x_l, X_1 \to X_1 \setminus X_l, B)] =$$

$$H(A) - \sum_{l=j}^{m-1} I(A; X_1 \to X_l \setminus X_{l+1} | x_l, X_1 \to X_1 \setminus X_l, B) \leq H(A).$$

$\blacksquare$

Now, let us define $r(p_i)$ such that $p_i \in X_{r(p_i)} \setminus X_{r(p_i)+1}$. Then,

$$\sum_{l=j}^{n} \sum_{p_i \in X_l} I(X_l \to p_i; x_l | X_l \to (X_1 \setminus X_l)) = \sum_{p_i \in X_j} \sum_{l=j}^{r(p_i)} I(X_l \to p_i; x_l | X_l \to (X_1 \setminus X_l)) =$$

$$\sum_{p_i \in X_j} \sum_{l=j}^{r(p_i)} I\left(\bigcup_{m=l}^{r(p_i)-1} ((X_m \setminus X_{m+1}) \to p_i) \cup (X_{r(p_i)} \to p_i); x_l | X_l \to X_1 \setminus X_l \right)$$

By the chain rule

$$= \sum_{p_i \in X_j} \sum_{l=j}^{r(p_i)} \left(\sum_{m=l}^{r(p_i)-1} I(X_m \setminus X_{m+1} \to p_i; x_l | X_l \to X_1 \setminus X_l, X_{m+1} \to p_i) + I((X_{r(p_i)} \to p_i); x_l | X_l \to X_1 \setminus X_l) \right) =$$

$$\sum_{p_i \in X_j} \left(\sum_{m=j}^{r(p_i)-1} \sum_{l=j}^{m} I(X_m \setminus X_{m+1} \to p_i; x_l | X_l \to X_1 \setminus X_l, X_{m+1} \to p_i) + \sum_{l=j}^{r(p_i)} I(X_{r(p_i)} \to p_i; x_l | X_l \to X_1 \setminus X_l) \right)$$

by Sublemma 1(we also seperate $p_1 \in X_n$ from other peers, and writing it in close form)

$$\leq \sum_{l=j}^{n} I(X_l \to p_1; x_l | X_l \to X_1 \setminus X_l) + \sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right)$$

Then

$$\sum_{l=1}^{j-1} H(x_l) + \sum_{l=j}^{n} I(X_l \to (X_1 \setminus X_l); x_l) + \sum_{l=j}^{n} \sum_{p_i \in X_l} I(X_l \to p_i; x_l | X_l \to (X_1 \setminus X_l)) \leq$$

$$\sum_{l=1}^{j-1} H(x_l) + \sum_{l=j}^{n} I(X_l \to (X_1 \setminus X_l); x_l) + \sum_{l=j}^{n} I(X_l \to p_1; x_l | X_l \to X_1 \setminus X_l) +$$

$$\sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) =$$

$$\sum_{l=1}^{j-1} H(x_l) + \sum_{l=j}^{n} [I(X_l \to (X_1 \setminus X_l); x_l) + I(X_l \to p_1; x_l | X_l \to X_1 \setminus X_l)] +$$

$$\sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) =$$

$$\sum_{l=1}^{j-1} H(x_l) + \sum_{l=j}^{n} I(X_l \to X_1 \setminus X_l, X_l \to p_1; x_l) +$$

$$\sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) =$$

$$\sum_{l=1}^{j-1} H(x_j) + \sum_{l=j}^{n} H(x_l) + \sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) =$$

$$\sum_{l=1}^{n} H(x_l) + \sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) =$$

$$H(X_1 \to p_1) + \sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) \le$$

$$H(X_j \to p_1) + H(X_1 \setminus X_j \to p_1) + \sum_{p_i \in X_j \setminus \{p_1\}} \left(\sum_{m=j}^{r(p_i)-1} H(X_m \setminus X_{m+1} \to p_i) + H(X_{r(p_i)} \to p_i) \right) \le$$

$$\sum_{p_l \in X_j} H(p_l \to p_1) + \sum_{p_l \in X_j} H(p_l \to X_1 \setminus X_j) + \sum_{p_l \in X_j} \sum_{p_i \in X_j \setminus \{p_1\}} H(p_l \to p_i) \le$$

$$\sum_{p_l \in X_j} \sum_{p_i \in X_1 \setminus X_j} H(p_l \to p_i) + \sum_{p_l \in X_j} \sum_{p_i \in X_j} H(p_l \to p_i) = \sum_{p_l \in X_j} \sum_{p_i \in X_1} H(p_l \to p_i) \le \sum_{p_l \in X_j} C_l$$

Last inequality is due to *property iii*. ∎

Proof of Lemma 6:
Let $A_1 = \{N_1, N_2, \ldots, N_{m-1}\}$ and $j_1 = min\{r : N_r = max(A_1)\}$. Then, iteratively define $A_i = A_{i-1} \setminus \{N_{j_{i-1}}, N_{j_{i-1}+1}, \ldots, N_{m-1}\}$ and $j_i = min\{r : N_r = max(A_i)\}$ for $i > 1$. Note that $\exists\ l \in \{1, 2, \ldots, m-1\}$ such that $j_l = 1$ since $|A_i|$ is strictly decreasing and so is the sequence $\{j_i\}$. Moreover, note that $m > j_1 > j_2 > \ldots > j_l = 1$ and $0 \ge N_{j_1} > N_{j_2} > \ldots > N_{j_l} = N_1$.

Sublemma 2.

$$N - N_1 \ge \frac{|X_M|N}{|X_M| - 1}$$

Proof: Using $\sum\limits_{p_i \in X_1} C_i + C_s - \sum\limits_{i=1}^{n} |X_i| L_i = -N_1$, we get

$$
\begin{aligned}
\frac{N}{|X_M| - 1} \quad &\leq \quad \frac{N_M}{|X_M| - 1} \\
&\leq \quad \sum_{j=1}^{h-1} \frac{N_{i_j} - N_{i_{j+1}}}{|X_{i_j}| - 1} + \frac{N_M}{|X_M| - 1} \\
&= \quad \sum_{j=1}^{h} \frac{N_{i_j} - N_{i_{j+1}}}{|X_{i_j}| - 1} \ \ since \ i_h = M \ and \ N_{i_{h+1}} = 0 \\
&\leq \quad \sum_{p_j \in X_1} C_j + C_s - \sum_{j=1}^{n} |X_j| L_j \\
&= \quad -N_1
\end{aligned}
$$

Hence, we get $\frac{N}{|X_M|-1} \leq -N_1$ which implies $N - N_1 \geq \frac{|X_M| N}{|X_M|-1}$, as desired. ∎

Identity 1. *If $j < t$,*

$$
\sum_{p_i \in X_j \setminus X_t} C_i = N_t - N_j + \sum_{i=j}^{t-1} L_i(|X_i| - 1)
$$

holds.

Proof: Note that we have $\sum_{p_i \in X_j} C_i = \sum_{i=j}^{n} |X_i| L_i + \sum_{i=1}^{j-1} L_i - C_s - N_j$, $\sum_{p_i \in X_t} C_i = \sum_{i=t}^{n} |X_i| L_i + \sum_{i=1}^{t-1} L_i - C_s - N_t$ and $\sum_{p_i \in X_j \setminus X_t} C_i = \sum_{p_i \in X_j} C_i - \sum_{p_i \in X_t} C_i$. Hence, we get

$$
\begin{aligned}
\sum_{p_i \in X_j \setminus X_t} C_i \quad &= \quad \left(\sum_{i=j}^{n} |X_i| L_i + \sum_{i=1}^{j-1} L_i - C_s - N_j\right) - \left(\sum_{i=t}^{n} |X_i| L_i + \sum_{i=1}^{t-1} L_i - C_s - N_t\right) \\
&= \quad N_t - N_j + \sum_{i=j}^{t-1} L_i(|X_i| - 1).
\end{aligned}
$$

This completes the proof of identity. ∎

We will give the proof of lemma in the following two complementary cases:

Case 1: $\frac{|X_M| N}{|X_M|-1} \leq N - N_{j_1}$ holds.

By Identity 1, we have

$$
\begin{aligned}
\sum_{p_i \in X_{j_1} \setminus X_m} C_i \quad &= \quad N_m - N_{j_1} + \sum_{i=j_1}^{m-1} L_i(|x_i| - 1) \\
&\geq \quad N - N_{j_1} \\
&\geq \quad \frac{|X_M| N}{|X_M| - 1}
\end{aligned} \tag{16}
$$

Hence, $\forall p_i \in X_{j_1} \setminus X_m \ \exists 0 \leq C_{iM} \leq C_i$ such that

$$
\sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} = \frac{|X_M| N}{|X_M| - 1}. \tag{17}
$$

Note that

$$
\sum_{p_i \in X_j \setminus X_m} C_{iM} \leq \sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} = \frac{|X_M| N}{|X_M| - 1} \leq N - N_{j_1} \leq N - N_j \tag{18}
$$

$\forall j \in \{j_1, j_1 + 1, \ldots, m - 1\}$ since $N_{j_1} \geq N_j$ $\forall j \in \{j_1, j_1 + 1, \ldots, m - 1\}$ by the choice of j_1. Now, we choose $C_{iM} = 0$, $\forall p_i \in X_1 \setminus X_{j_1}$. Hence, usign $C_{iM} = 0$, $\forall p_i \in X_1 \setminus X_{j_1}$ we get,

$$\sum_{p_i \in X_j \setminus X_m} C_{iM} = \sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} = \frac{|X_M| N}{|X_M| - 1} \leq N - N_{j_1} \leq N - N_j \tag{19}$$

$\forall j \in \{1, 2, \ldots, j_1 - 1\}$ since $N_{j_1} \geq N_j$ $\forall j \in \{1, 2, \ldots, j_1 - 1\}$ by the choice of j_1. Hence, as can be seen from (17), (18) and (19), this particular choice of C_{iM} $\forall p_i \in X_1 \setminus X_m$ is exactly as desired. This completes proof of *Case 1*.

Case 2: $\frac{|X_M| N}{|X_M| - 1} > N - N_{j_1}$ holds.
Substituting $N_1 = N_{j_l}$ in Sublemma, we get:

$$N - N_{j_l} \geq \frac{|X_M| N}{|X_M| - 1} > N - N_{j_1}$$

which implies that $\exists t \in \{1, 2, \ldots, l\}$ such that

$$N - N_{j_{t+1}} \geq \frac{|X_M| N}{|X_M| - 1} > N - N_{j_t} \tag{20}$$

holds. Now, we will give an iterative method for the choice of C_{iM}, $\forall p_i \in X_1 \setminus X_m$ satisfying required constraints as follows:
First Step: Choice of C_{iM}, $\forall p_i \in X_{j_1} \setminus X_m$
By (16), we have $\sum_{p_i \in X_{j_1} \setminus X_m} C_i \geq N - N_{j_1}$. Hence, $\forall p_i \in X_{j_1} \setminus X_m$ $\exists C_{iM}$ with $0 \leq C_{iM} \leq C_i$ such that

$$\sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} = N - N_{j_1}. \tag{21}$$

Note that

$$\sum_{p_i \in X_j \setminus X_m} C_{iM} \leq \sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} = N - N_{j_1} \leq N - N_j \tag{22}$$

$\forall j \in \{j_1, j_1 + 1, \ldots, m - 1\}$ since $N_{j_1} \geq N_j$ $\forall j \in \{j_1, j_1 + 1, \ldots, m - 1\}$ by the choice of j_1.
r-th step for $1 < r \leq t$: Choice of C_{iM}, $\forall p_i \in X_{j_r} \setminus X_{j_{r-1}}$.
By Identity 1, we have

$$\begin{aligned}
\sum_{p_i \in X_{j_r} \setminus X_{j_{r-1}}} C_i &= N_{j_{r-1}} - N_{j_r} + \sum_{i=j_r}^{j_{r-1}} L_i(|x_i| - 1) \\
&\geq N_{j_{r-1}} - N_{j_r}
\end{aligned}$$

Hence, $\forall p_i \in X_{j_r} \setminus X_{j_{r-1}}$ $\exists C_{iM}$ with $0 \leq C_{iM} \leq C_i$ such that

$$\sum_{p_i \in X_{j_r} \setminus X_{j_{r-1}}} C_{iM} = N_{j_{r-1}} - N_{j_r}. \tag{23}$$

Using (21) and (23), $\forall r \in \{2, 3, \ldots, t\}$ and $\forall j \in \{j_r, j_r + 1, \ldots, j_{r-1} + 1\}$ we have

$$\begin{aligned}
\sum_{p_i \in X_j \setminus X_m} C_{iM} &\leq \sum_{p_i \in X_{j_r} \setminus X_m} C_{iM} \\
&= \sum_{p_i \in X_{j_1} \setminus X_m} C_{iM} + \sum_{q=2}^{r} \left(\sum_{p_i \in X_{j_q} \setminus X_{j_{q-1}}} C_{iM} \right) \\
&= (N - N_{j_1}) + \sum_{q=2}^{r} (N_{j_{q-1}} - N_{j_q}) \\
&= N - N_{j_r} \\
&\leq N - N_j \tag{24}
\end{aligned}$$

since $N_{j_r} = max\{N_1, N_2, \ldots, N_{j_{r-1}+1}\}$ by the choice of j_r.

$(t+1)$-*th step*: Choice of C_{iM}, $\forall p_i \in X_{j_{t+1}} \setminus X_{j_t}$ and $\forall p_i \in X_1 \setminus X_{j_{t+1}}$, separately.
Remember we have (20) by the choice of t. Using Identity 1 and (20), we get

$$
\begin{aligned}
\sum_{p_i \in X_{j_{t+1}} \setminus X_{j_t}} C_{iM} &\geq N_{j_t} - N_{j_{t+1}} \\
&= (N - N_{j_{t+1}}) - (N - N_{j_t}) \\
&\geq \frac{|X_M|N}{|X_M| - 1} - (N - N_{j_t})
\end{aligned}
$$

Hence $\forall p_i \in X_{j_{t+1}} \setminus X_{j_t}$, $\exists C_{iM}$ with $0 \leq C_{iM} \leq C_i$ such that

$$
\sum_{p_i \in X_{j_{t+1}} \setminus X_{j_t}} C_{iM} = \frac{|X_M|N}{|X_M| - 1} - (N - N_{j_t}). \tag{25}
$$

Using (25), we get

$$
\begin{aligned}
\sum_{p_i \in X_{j_{t+1}} \setminus X_m} C_{iM} &= \sum_{p_i \in X_{j_{t+1}} \setminus X_{j_t}} C_{iM} + \sum_{p_i \in X_{j_t} \setminus X_m} C_{iM} \\
&= \frac{|X_M|N}{|X_M| - 1} - (N - N_{j_t}) + (N - N_{j_t}) \\
&= \frac{|X_M|N}{|X_M| - 1} \tag{26}
\end{aligned}
$$

Now, choose $C_{iM} = 0$, $\forall p_i \in X_1 \setminus X_{j_{t+1}}$. Using (26) and the choice of $C_{iM} = 0$, $\forall p_i \in X_1 \setminus X_{j_{t+1}}$, we get for any $j \in \{1, 2, \ldots, j_{t+1} - 1\}$ that

$$
\begin{aligned}
\sum_{p_i \in X_j \setminus X_m} C_{iM} &= \sum_{p_i \in X_j \setminus X_{j_{t+1}}} C_{iM} + \sum_{p_i \in X_{j_{t+1}} \setminus X_m} C_{iM} \\
&= 0 + \frac{|X_M|N}{|X_M| - 1} \\
&= \frac{|X_M|N}{|X_M| - 1} \tag{27}
\end{aligned}
$$

Using (27), we get for any $j \in \{1, 2, \ldots, j_t - 1\}$ that

$$
\begin{aligned}
\sum_{p_i \in X_j \setminus X_m} C_{iM} &\leq \sum_{p_i \in X_{j_{t+1}} \setminus X_m} C_{iM} \\
&= \frac{|X_M|N}{|X_M| - 1} \\
&\leq N - N_{j_{t+1}} \\
&\leq N - N_j \tag{28}
\end{aligned}
$$

since $N_{j_{t+1}} = max\{N_1, N_2, \ldots, N_{j_t-1}\}$ by the choice of j_{t+1}. Putting $j = 1$ in (27) gives

$$
\sum_{p_i \in X_1 \setminus X_m} C_{iM} = \frac{|X_M|N}{|X_M| - 1} \tag{29}
$$

Finally, considering (22), (24) and (28), this particular choice of C_{iM} for each peer $p_i \in X_1 \setminus X_m$ satisfies

$$
\sum_{p_i \in X_j \setminus X_m} C_{iM} \leq N - N_j
$$

$\forall j \in \{1, 2, \ldots, m - 1\}$ and by (29)

$$
\sum_{p_i \in X_1 \setminus X_m} C_{iM} = \frac{|X_M|N}{|X_M| - 1},
$$

as desired. This completes the proof of Lemma 6. ∎

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